Non-Spherical Newtonian Gravitational Collapse: a Set of Exact Nonlinear Closed–Form Solutions

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Abstract

We give a set of exact nonlinear closed–form solutions for the non-spherical collapse of pressure-less matter in Newtonian gravity, and indicate their possible cosmological applications.

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1 Introduction

The Zeldovich approximation [1], of decades ago, serves as a reliable basis for modern studies of the large-scale cosmic structure formation (see the classic paper [2], recent works [3], and the references therein). It stemmed from an elegant exact nonlinear solution for the Newtonian anisotropic motion of dust-like gravitating matter, which describes a growth of initially weak adiabatic perturbations due to gravitational instability in an expanding universe.

We re-visit the Zeldovich solution and its generalizations [4, 5] to reveal their new features emerging under the time-reversal transformation (TrT), $t \longrightarrow -t$. To start with, we note that Friedmann's general–relativistic cosmological solution is

known to have a Newtonian analog corresponding to a uniform isotropic expansion of a perfect pressure-less gravitating fluid ('dust'). After the TrT, it turns to the solution describing the isotropic gravitational collapse (parabolic motion):

$$\vec{x}(t,\vec{\xi}) = (t_0 - t)^{2/3}\vec{\xi}; \qquad \rho(t) = [6\pi G(t_0 - t)^2]^{-1} \; . \tag{1}$$

Here $\vec{x}$ is the Euler, and $\vec{\xi}$ is the Lagrange coordinate, G is the Newtonian gravitation constant, $\rho(t)$ is the matter density, and t_0 is the collapse time, when the scale factor $a(t) = (t_0 - t)^{2/3}$ becomes zero and the density goes to infinity. Solution (1) applies to uniform spheres of a finite total mass and any initial radius.

In this paper we give a set of solutions for the non-spherical (anisotropic) collapse using the time reversal of expansion solutions found in [1, 4, 5], and discuss their properties. The main reason for this is that in modern cosmology three types of building blocks are recognized forming the large-scale Cosmic Web. Those are rich spherical clusters of galaxies, flat superclusters ('pan-cakes'), and elongated filaments. With all the necessary reservations, the formation and evolution of spherical clusters can hopefully be described by the asymptotically spherical collapse solution (4). Superclusters and filaments are seen as highly oblate and highly prolate structures, respectively. Their evolution may be described by the solutions (6),(7) and (10),(11) containing the growing mode, assuming that they correctly expose the basic features of a real finite mass collapse with the same dynamical asymptotics. Generalized formulas (13) add dark energy to the picture.

One can construct the collapse solutions directly, without the TrT; this explains the powers of $\tau = t_0 - t$ involved in them. Assume that the flow (1) is perturbed in one direction by a term with separated variables:

$$x_1 = (\tau)^{2/3}\xi_1 + g(\tau)f(\xi_1); \qquad x_{2,3} = (\tau)^{2/3}\xi_{2,3} \; ; \tag{2}$$

functions $g(\tau)$ and $f(\xi_1)$ are so far arbitrary. By repeatedly differentiating these expressions one obtains the accelerations $\partial^2 x_n/\partial t^2$, $n = 1,2,3$; according to the

equations of motion, they are the negative of the respective components of the gravitational potential gradient. The divergence of the gradient produces the Laplacian of the potential; divided by $4\pi G$, it gives, by the Newton law of gravity, the first, 'gravitational', formula for the matter density. The other one comes from the mass conservation, i.e., the continuity equation; the two densities coincide when

$$g^{''} - (4/9\tau^2)g = 0 \; ; \qquad (3)$$

linearly independent solutions of this equation are $\tau^{4/3}$ and $\tau^{-1/3}$. The first one corresponds to the solution (4) of sec. 2, the second gives the solution (6), sec. 3. A linear combination, $\tau^{4/3}f(\xi_1)+\tau^{-1/3}F(\xi_1)$, turns out also possible in the expression (2) for x_1, allowing for the solution (10) of sec. 4.

2 Collapse with decaying asphericity

The well–known Zeldovich's 'pan-cake' solution [1] for planar motion of pressure-free matter takes the form, after the TrT :

$$x_1(t,\xi_1) = (t_0-t)^{2/3}\xi_1 + (t_0-t)^{4/3}f(\xi_1), \qquad x_{2,3}(t,\xi_{2,3}) = (t_0-t)^{2/3}\xi_{2,3} \; ; \qquad (4)$$

$$\rho(t,\xi_1) = \frac{1}{6\pi G(t_0-t)^2}\,\frac{1}{1+(t_0-t)^{2/3}f'(\xi_1)}, \qquad f^{'}(\xi_1) = \frac{df}{d\xi_1}\;. \qquad (5)$$

The deviation in x_1 from the spherically symmetric motion (1) goes to zero when $t \to t_0 - 0$, becoming negligible against the background (a decaying mode). So formulas (4),(5) describe an initially non-spherical collapse which undergoes spherization and becomes completely spherical in the end.

The density (5) is non-singular until the collapse when $1+(t_0-t)^{2/3}f^{'}(\xi_1) > 0$ for all ξ_1, guaranteeing also a unique inverse, $\vec{\xi} = \vec{\xi}(t,\vec{x})$, of the law of motion $\vec{x} = \vec{x}(t,\vec{\xi})$, as required. The inequality, and thus the solution, is apparently valid for a *finite* period of time, $t_i < t < t_0$, if $f^{'}(\xi_1) \geq -(t_0-t_i)^{-2/3}$. For $t_i = -\infty$, function $f(\xi_1)$ is non–decreasing, $f^{'}(\xi_1) \geq 0$, so the solution holds on the whole semi–axis $-\infty < t < t_0$.

3 Collapse with growing asphericity

A counterpart to the solution (4),(5) comes from [4]: using the TrT, one finds:

$$x_1(t,\xi_1) = (t_0 - t)^{2/3}\xi_1 + (t_0 - t)^{-1/3}F(\xi_1), \qquad x_{2,3}(t,\xi_{2,3}) = (t_0 - t)^{2/3}\xi_{2,3}\ ; \quad (6)$$

$$\rho(t,\xi_1) = \frac{1}{6\pi G(t_0 - t)^2}\,\frac{1}{1 + (t_0 - t)^{-1}F'(\xi_1)}\ . \quad (7)$$

The deviation from the spherical symmetry here increases with the time (growing mode); initially ($t = -\infty$) the flow is entirely spherical. The density is non–singular until $t = t_0$ if and only if $F^{'}(\xi_1) \geq 0$. Unlike the previous two cases (1) and (4), near the collapse it is inversely proportional to the first, instead of the second, power of $(t_0 - t)$ and, generically, remains non-uniform, $\rho \sim \left[6\pi G(t_0 - t)F^{'}(\xi_1)\right]^{-1}$; instead of a point, the dust collapses to the x_1–axis.

If the last inequality is violated, and the minimum of $F^{'}(\xi_1)$ is at a single point,

$$\min_{\xi_1} F^{'}(\xi_1) = F^{'}(\xi_1^*) = -t_* < 0\ , \quad (8)$$

then a *local* collapse occurs at $t = t_0 - t_*$ before the global one at $t = t_0$: the density becomes infinite at the plane $\xi_1 = \xi_1^*$, or $x_1 = t_*^{2/3}\xi_1^* + t_*^{-1/3}F(\xi_1^*)$. Note that $\rho(t,\xi_1^*)$ is maximum before the collapse; at the collapse, $t = t_0 - t_*$, the density at all other points $\xi_1 \neq \xi_1^*$ remains finite and non-zero.

If the same minimum (8) is achieved at a number of discrete ξ_1–points, then the 'early' collapse happens at all the corresponding planes. If Eq. (8) holds for some ξ_1– interval,

$$F(\xi_1) = -t_*\xi_1, \qquad \alpha < \xi_1 < \beta\ , \quad (9)$$

then $x_1(t,\xi_1) = (t_0 - t)^{-1/3}(t_0 - t_* - t)\xi_1$, $x_1(t_0 - t^*, \xi_1) = 0$ for $\alpha < \xi_1 < \beta$, and this whole ξ_1–layer collapses to a single plane $x_1 = 0$. Ultimately, when Eq. (8) is valid for *all* ξ_1, i.e., $F(\xi_1) \equiv -t_*\xi_1$, the early collapse becomes *global*: the density $\rho(t,\xi_1) = [6\pi G(t_0 - t)(t_0 - t_* - t)]^{-1}$ is uniform, so the whole space collapses. Still, the flow remains anisotropic and the dust goes to the plane $x_1 = 0$, and not to a point.

4 General solution

In paper [4], a solution containing both modes was also found; its time reversal is:

$$x_1(t, \xi_1) = (t_0 - t)^{2/3}\xi_1 + (t_0 - t)^{4/3} f(\xi_1) + (t_0 - t)^{-1/3} F(\xi_1) \; ; \tag{10}$$

$$\rho(t, \xi_1) = \frac{1}{6\pi G(t_0 - t)^2} \frac{1}{1 + (t_0 - t)^{2/3} f'(\xi_1) + (t_0 - t)^{-1} F'(\xi_1)} \tag{11}$$

($x_{2,3}$ are as before). It is general in a sense that the two free functions $f(\xi_1)$ and $F(\xi_1)$ allow one to meet any initial conditions for both the x_1-coordinate and x_1-velocity. The solution (10),(11) is valid at all times for $f^{'}(\xi_1) \geq 0$, $F^{'}(\xi_1) \geq 0$; other cases are analyzed as above. Of course, the decaying mode plays no role near the collapse, which is thus either an early ($t < t_0$) planar, or a global axial one at $t = t_0$.

Note that solutions describing the expansion with an anisotropic flow and uniform density $\rho(t) = [6\pi G(t - t_0)]^{-2}$ were studied in paper [6]. Their time reversals can add to the set of non–spherical collapse solutions obtained here.

5 Anisotropic collapse of a finite mass

All the solutions of secs. 2 - 4 apply to circular cylinders of a finite mass with the symmetry axis along x_1, of an arbitrary initial radius and height, whose density is uniform in both the radial and azimuthal directions. Such a cylinder is specified by the following bounded set of Lagrange variables:

$$\alpha < \xi_1 < \beta, \quad \sqrt{\xi_2^2 + \xi_3^2} < \gamma; \qquad -\infty < \alpha < \beta < \infty, \quad 0 < \gamma < \infty \, .$$

The decaying–mode solution of sec. 2 describes the collapse of a cylinder to a point. Two other solutions, of secs. 3 and 4, containing the growing mode, end up with an early ($t < t_0$) collapse to a disk (or a number of disks) perpendicular to the x_1 axis, when $F^{'}(\xi_1) < 0$ for some values of ξ_1 in the above range. Otherwise $F^{'}(\xi_1) > 0$ for all ξ_1, and the cylinder collapses at $t = t_0$ to a part of the x_1–axis. Depending

on the behavior of $F(\xi_1)$, it can be either the whole axis, or semi–axis, or a finite segment moving to infinity when $t \to t_0 - 0$. So the outcome of the collapse process is determined by the initial conditions in x_1–direction, if other initial conditions provide for the axial symmetry. Namely, at an initial moment of time $t = t_i$ the velocities should be consistent with the coordinates, $v_{2,3} = -(2/3)[x_{2,3}/(t_0 - t_i)]$, which relations hold then on all the way to the collapse.

6 Collapse on the dark energy background

The original Friedmann cosmological solution includes the cosmological constant Λ, which may be zero or non-zero. The time-reversed isotropic flow with $\Lambda > 0$ is:

$$\vec{x}(t,\vec{\xi}) = a(t)\vec{\xi}, \quad \rho(t) \propto [a(t)]^{-3}; \qquad a(t) \propto \sinh^{2/3}\left[(3/2)(\sqrt{3\Lambda})(t_0 - t)\right] \; . \quad (12)$$

The corresponding non-spherical solution of paper [5] after the TrT becomes:

$$x_1(t,\xi_1) = a(t)\xi_1 + \phi(t)f(\xi_1) + \Phi(t)F(\xi_1), \qquad x_{2,3}(t,\xi_{2,3}) = a(t)\xi_{2,3} \; . \quad (13)$$

Functions $\phi(t)$ and $\Phi(t)$ giving the time dependence of the decaying and growing mode are found in [5]. Solution (13) is general in the same sense as in the case (10).

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